FINITE AUTOMORPHISMS OF AFFINE n-SPACE

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Abstract. It is still an open question whether or not there exist polynomial automorphisms of finite order of complex affine n-space which cannot be linearized, i.e., which are not conjugate to linear automorphisms. The second author gave the first examples of non-linearizable actions of positive dimensional groups, and Masuda and Petrie did the same for finite groups.

These examples were all obtained from non-trivial G-vector bundles on representation spaces using ideas of Bass and Haboush. So far, this approach has failed for commutative groups and in particular for automorphisms of finite order. The reason is given by a recent theorem due to Masuda, Moser-Jauslin and Petrie showing that for a commutative reductive group G every G-vector bundle on a representation space of G is trivial.

The aim of this report is to give an introduction to the subject, to describe some basic results and to present a short proof of the theorem of Masuda, Moser-Jauslin and Petrie from a different perspective (cf. [KS92]).

§1. Introduction

Let $\mathbb{A}^n$ denote affine n-space over the field $\mathbb{C}$ of complex numbers. In this note we want to discuss the following question:

Problem. *Is every finite order polynomial automorphism of affine n-space $\mathbb{A}^n$ conjugate to a linear automorphism?*

The case of the affine line $\mathbb{A}^1$ is easy: Every polynomial automorphism φ is an affine transformation, i.e., $\varphi(x) = ax + b$ $(a, b \in \mathbb{C},\ a \neq 0)$. If φ is not a translation, then it has a fixed point p, and the conjugate automorphism $t_p^{-1} \circ \varphi \circ t_p$ is linear. (Here t_p denotes the translation $x \mapsto x + p$.)

For the affine plane $\mathbb{A}^2$ the answer is also positive, but this is already a difficult theorem due to Suzuki [Su74]. It is also a consequence of the amalgamated product structure of the automorphism group of $\mathbb{A}^2$. We will discuss this in the next section.

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The general question is completely open. In a recent paper Asanuma gives an example of a non-linearizable automorphism of finite order in positive characteristic (see [As94, Remark 3.4]).

A first basic result concerning fixed points of polynomial automorphisms is given in the proposition below. It follows from a general result of Verdier ([Ve73]). A more direct approach was given by Petrie and Randall [PR86] by showing the existence of an equivariant compactification.

Proposition 1. *Every finite order polynomial automorphisms of $\mathbb{A}^n$ has a fixed point.*

Remark. We do not know if every finite group of automorphisms of $\mathbb{A}^n$ has fixed points.

§2. The amalgamated product structure of $\operatorname{Aut}\mathbb{A}^2$

Our further discussion requires some notation. Recall that a polynomial map

$$\varphi = (\varphi_1, \varphi_2, \dots, \varphi_n) \colon \mathbb{A}^n \to \mathbb{A}^n, \quad \varphi_i \in \mathbb{C}[x_1, x_2, \dots, x_n]$$

is an *automorphism* (i.e., has a polynomial inverse) if and only if it is bijective (cf. [Kr85, Lemma II.3.4]). We denote by $\operatorname{Aut}\mathbb{A}^n$ the *group of polynomial automorphisms* of $\mathbb{A}^n$. There are two important subgroups of $\operatorname{Aut}\mathbb{A}^n$, the subgroup of *affine transformations*

$$\mathfrak{A}_n := \{\varphi = (\varphi_1, \dots, \varphi_n) \in \operatorname{Aut}\mathbb{A}^n \mid \varphi_i \text{ linear for all } i\}$$

and the *Jonquière subgroup*

$$\mathfrak{J}_n := \{\varphi = (\varphi_1, \dots, \varphi_n) \in \operatorname{Aut}\mathbb{A}^n \mid \varphi_i \in \mathbb{C}[x_1, \dots, x_i], i = 1, \dots, n\}.$$

The intersection $\mathfrak{B}_n := \mathfrak{A}_n \cap \mathfrak{J}_n$ is formed by the lower triangular affine transformations. The theorem below is the fundamental result about the structure of the automorphism group of $\mathbb{A}^2$. It is due to van der Kulk [vK53] and has been reproved with different methods by several authors (see e.g. [GD75]).

Theorem 1. *The group $\operatorname{Aut}\mathbb{A}^2$ is the amalgamated free product of $\mathfrak{A}_2$ and $\mathfrak{J}_2$ over their intersection $\mathfrak{B}_2$:*

$$\operatorname{Aut}\mathbb{A}^2 = \mathfrak{A}_2 \star_{\mathfrak{B}_2} \mathfrak{J}_2.$$

This amalgamated product structure means that every element φ of $\operatorname{Aut}\mathbb{A}^2$ can be written in the form

$$(*) \qquad \varphi = \cdots \alpha_i \gamma_i \alpha_{i+1} \gamma_{i+1} \cdots \quad \text{where} \quad \alpha_i \in \mathfrak{A}_2, \gamma_i \in \mathfrak{J}_2$$

and that this representation is unique up to the obvious relations $\alpha\gamma = (\alpha\beta^{-1})(\beta\gamma)$ and $\gamma\alpha = (\gamma\beta^{-1})(\beta\alpha)$ for $\beta \in \mathfrak{B}_2$. The number of elements in the expression $(*)$ is called the *length* of φ.

From this theorem it is easy to see that every automorphism φ of finite order of $\mathbb{A}^2$ is conjugate to a linear one. In fact, let us assume that φ has order s and that the expression $(*)$ starts with $\alpha_1 \in \mathfrak{A}_2$. (The other case can be treated in a similar way.) Then $(*)$ has to end with an element of $\mathfrak{A}_2$ since otherwise the length of $\varphi^s =$ identity would be s times the length of φ. Thus $\varphi = \alpha_1\gamma_1 \cdots \alpha_{n-1}\gamma_{n-1}\alpha_n$ and

$$\varphi^2 = \alpha_1\gamma_1 \cdots \alpha_{n-1}\gamma_{n-1}\alpha_n\alpha_1\gamma_1 \cdots \alpha_{n-1}\gamma_{n-1}\alpha_n.$$

As above, arguing with the length of φ^s, we find that $\alpha_n\alpha_1 \in \mathfrak{B}_2$. Hence, the conjugate element

$$\alpha_1^{-1}\varphi\alpha_1 = \gamma_1\alpha_2 \cdots \alpha_{n-1}\gamma_{n-1}\alpha_n\alpha_1 = \gamma_1\alpha_2 \cdots \alpha_{n-1}\gamma'_{n-1}$$

has shorter length than φ. Now we see by induction that φ is conjugate to an element of $\mathfrak{A}_2$ or $\mathfrak{J}_2$, and from this the claim follows immediately. □

The above is a special case of the following result due to SERRE (see [Se80]).

Proposition 2. *A subgroup of bounded length of an amalgamated product is conjugate to a subgroup of one of the factors.*

WRIGHT [Wr79] points out that every *algebraic* subgroup of $\operatorname{Aut}\mathbb{A}^2$ is of bounded length. Combining all this we obtain the following result about algebraic group actions on the affine plane $\mathbb{A}^2$. It was first formulated by KAMBAYASHI in [Ka79]:

Theorem 2. *Every algebraic subgroup $G \subset \operatorname{Aut}\mathbb{A}^2$ is conjugate to a subgroup of $\mathfrak{A}_2$ or $\mathfrak{J}_2$. In particular, every reductive subgroup $G \subset \operatorname{Aut}\mathbb{A}^2$ is linearizable.*

An amalgamated product structure for $\operatorname{Aut}\mathbb{A}^n$ as in Theorem 1 does not exist in general. This is shown by the following easy example communicated to us by JACQUES ALEV.

EXAMPLE. Let $\sigma_1, \sigma_2, \tau \in \operatorname{Aut}\mathbb{A}^3$ be defined in the following way:

$$\sigma_1(x,y,z) := (x,y,z+x^2), \quad \sigma_2(x,y,z) := (x,y+x^2,z) \quad \tau(x,y,z) := (x,z,y).$$

Then

$$\sigma_1,\sigma_2 \in \mathfrak{J}_3 \setminus \mathfrak{B}_3, \quad \tau \in \mathfrak{A}_3 \setminus \mathfrak{B}_3, \quad \text{and} \quad \tau\sigma_1\tau = \sigma_2.$$

In [Ba84] BASS gives an example of a subgroup of $\operatorname{Aut}\mathbb{A}^3$ isomorphic to the additive group $\mathbb{C}^+$ which cannot be *triangularized*, i.e., which is not conjugate to a subgroup of $\mathfrak{J}_3$. This shows that the first part of Theorem 2 above does not hold in dimension $n > 2$. We will see in the next paragraph that in dimension $n \geq 4$ the second part also fails to be true. However, there are some positive results in low dimension due to POPOV, PANYUSHEV and the first author (see [KP85], [Pa84], cf. [KS92]):

Theorem 3. *Every (connected) semisimple subgroup of* $\operatorname{Aut}\mathbb{A}^n$, $n \leq 4$, *is linearizable.*

It is still an open question whether actions of the multiplicative group $\mathbb{C}^*$ or of finite groups on $\mathbb{A}^3$ are linearizable. We refer to [KR86,89a,89b] and [Kr90] for some interesting results in this direction.

§3. Non-linearizable group actions and G-vector bundles

In 1989 the second author gave the following example of a non-linearizable subgroup $\mathrm{O}_2 \subset \operatorname{Aut}\mathbb{A}^4$ ([Sch89], see [KS92]). Here O_2 denotes the usual complex orthogonal group, i.e., the semi-direct product of the multiplicative group $\mathbb{C}^*$ with $\mathbb{Z}/2$, and the embedding into $\operatorname{Aut}\mathbb{A}^4$ is given in the following way: An element $t \in \mathbb{C}^*$ acts by

$$t\left(\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} x \\ y \end{bmatrix}\right) = \left(\begin{bmatrix} t^2 a \\ t^{-2} b \end{bmatrix}, \begin{bmatrix} t^3 x \\ t^{-3} y \end{bmatrix}\right),$$

and the non-trivial element $\sigma \in \mathbb{Z}/2$ by

$$\sigma\left(\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} x \\ y \end{bmatrix}\right) = \left(\begin{bmatrix} b \\ a \end{bmatrix}, \begin{bmatrix} 1 + ab + (ab)^2 & -b^3 \\ a^3 & 1 - ab \end{bmatrix}\begin{bmatrix} y \\ x \end{bmatrix}\right).$$

It is not obvious at all that this subgroup cannot be linearized, i.e., conjugated into $\mathrm{GL}_4(\mathbb{C}) \subset \operatorname{Aut}\mathbb{A}^4$.

The general *Linearization Problem* is studied in great detail in the work [KS92] of the authors which was strongly influenced by original ideas of Domingo Luna. It contains an intensive discussion of the basic material and gives many more examples of the kind above. The examples in [Sch89] initiated an exciting development (see [Kn91], [KS92], [MP91], [MMP91]). In particular, it was shown by Knop in [Kn91] that non-linearizable actions exist for all connected semi-simple groups. Masuda and Petrie showed that many of the examples even produce *families of inequivalent actions.*

In the example above, the action of $\mathbb{C}^*$ on $\mathbb{A}^4$ is linear and the projection

$$p\colon \mathbb{A}^4 \longrightarrow \mathbb{A}^2, \quad \left(\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} x \\ y \end{bmatrix}\right) \mapsto \left(\begin{bmatrix} a \\ b \end{bmatrix}\right)$$

is O_2 equivariant. Let us denote by V_m the 2-dimensional irreducible representation of O_2 with weights m and $-m$, i.e.,

$$t\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} t^m a \\ t^{-m} b \end{bmatrix} \ (t \in \mathbb{C}^*) \quad \text{and} \quad \sigma\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} b \\ a \end{bmatrix}.$$

Then $\mathbb{A}^4$ can be considered as a O_2-*vector bundle* over the representation space V_2 with zero fiber $p^{-1}(0)$ isomorphic to V_3 in the sense of the following definition (see [Kr89] for basic material about G-vector bundles):

DEFINITION. Let X be a variety with an action of an algebraic group G. A *G-vector bundle over X* is an algebraic vector bundle $p\colon \mathcal{V} \to X$ together with an action of G on $\mathcal{V}$ such that the projection p is equivariant and the action is linear on fibers.

In particular, if $x \in X$ is a fixed point then the fiber $\mathcal{V}_x := p^{-1}(x)$ is G-stable and the action on $\mathcal{V}_x$ is linear. A G-vector bundle of the form $\Theta_V := V \times X \xrightarrow{\text{pr}} X$ where V is a representation space of G is called a *trivial* G-vector bundle.

It was a basic idea of BASS and HABOUSH to consider G-vector bundles over representation spaces and to try to produce in this way non-linearizable actions ([BH87], cf. [Kr89]). Of course, they used the fundamental fact that the total space of every (algebraic) vector bundle over an affine space is again an affine space, due to the positive solution of the SERRE-Problem by QUILLEN and SUSLIN.

Before returning to our original problem (formulated in the introduction) we want to describe some of the basic general results in this context. Let G be a *reductive algebraic group*, i.e., an algebraic group whose finite dimensional representations are all completely reducible. Typical examples are GL_n, SL_n, the classical groups, the tori $(\mathbb{C}^*)^n$ and all finite groups. For these groups BASS and HABOUSH proved the following ([BH87]):

Proposition 3. *Let W be a finite dimensional representation space of the reductive group G. Then every G-vector bundle $\mathcal{V}$ over W is stably trivial, i.e., $\mathcal{V} \oplus \Theta_V$ is trivial for a suitable representation space V of G.*

This result is the basic ingredient in the work of MASUDA, MOSER-JAUSLIN and PETRIE ([MP91], [MMP91]). They study, for fixed representation spaces W, F and V, the set $\mathcal{V}_G(W, F; V)$ of all isomorphism classes of G-vector bundles on W with zero fiber F which are trivialized by Θ_V, and they construct an invariant for this set. In many cases this invariant can be explicitely calculated and gives raise to new examples of non-trivial G-vector bundles and non-linearizable actions. In particular, they discovered the first non-trivial G-vector bundles and non-linearizable actions for certain finite groups G, e.g., for dihedral groups of order $8m$ ($m \geq 3$). In several cases, their examples are restrictions to dihedral subgroups of the O_2 examples of [Sch89].

Our discussion above begs the question: How does one obtain non-linearizable actions from non-trivial G-vector bundles? We finish this section by describing some results in this direction.

Proposition 4. *Let W be a representation space of the reductive group G and let $\mathcal{V}$ be a G-vector bundle on W.*

(1) ([BH87]) *Assume that $\mathcal{V} \oplus \Theta_W$ in non-trivial. Then the G-action on $\mathcal{V}$ is non-linearizable.*

(2) ([MP91], [Kr89]) *If $\mathcal{V}$ is non-trivial then the action of $G \times H$ on $\mathcal{V}$ is non-linearizable for every non-trivial subgroup $H \subset \mathbb{C}^*$ acting by scalar multiplication on the fibers of $\mathcal{V}$.*

(3) ([MP91]) *If $\mathcal{V}$ is non-trivial and if there is a subgroup $H \subset G$ which acts trivially on W and such that $\mathcal{V}_0^H = \{0\}$, then the G-action on $\mathcal{V}$ is non-linearizable.*

As an application let us look again at the O_2-vector bundle $\mathcal{V} \to V_2$ from the example above. Then [Sch89] shows that $\mathcal{V} \oplus \Theta_V$ is non-trivial which gives the non-linearizability of the action by item (1) of the proposition above. One could also apply (3) setting $H := \{\pm 1\} \subset \mathbb{C}^*$.

§4. The equivariant Serre-Problem for abelian groups

We have already remarked above that all examples of non-linearizable actions on affine n-space known to date were obtained from non-trivial G-vector bundles on representation spaces. On the other hand, there are no known examples of non-linearizable actions of commutative (reductive) groups. All attempts via G-vector bundles have failed. The reason is given by the following very pretty result due to Masuda, Moser-Jauslin and Petrie [MMP94]. A special case of it was obtained earlier by Moser-Jauslin [Mo93] and independently and with different methods by DeConcini and Fagnani [DF94].

Theorem 4 (Masuda, Moser-Jauslin, Petrie)**.** *Let G be a commutative reductive group and let W be a finite dimensional representation space of G. Then every G-vector bundle on W is trivial.*

In the rest of this section, we present a proof in the spirit of [KS92], that is, we view vector bundles as glueings of trivial bundles over open covers of W. Our proof, however, follows from the ideas and results in [MMP94]. At the end, we give a slight extension of the theorem to actions of connected reductive groups.

Proof. By adding, if necessary, a direct summand we can assume that the representation of G on W is faithful. From now on, $\mathcal{V}$ denotes a fixed G-vector bundle over W. We denote its zero fiber $\mathcal{V}_0$ by V. Since G is reductive and commutative there is a basis $w_1, w_2, \dots, w_n$ of W consisting of G-eigenvectors, i.e., for all $g \in G$ we have $gw_i = \chi_i(g) \cdot w_i$ where χ_i is a suitable character of G. We denote by $x_1, x_2, \dots, x_n$ the dual basis to $w_1, w_2, \dots, w_n$ so that $\mathcal{O}(W) = \mathbb{C}[x_1, x_2, \dots, x_n]$. The weight of x_i is $-\chi_i$ and the hyperplanes $H_i := \{x_i = 0\}$ are all G-stable. We also fix a basis of eigenvectors $v_1, \dots, v_m$ of V, where v_i has character λ_i.

Claim 1. Every G-vector bundle on $W' := W \setminus \bigcup_{i=1}^n H_i$ is trivial.

Clearly, the action of G on W' is free. More precisely, the quotient $W' \to W'/\!\!/G$ is a principal G-bundle. Now we use a fundamental result due to Gubeladze

[Gu88]. It proves a conjecture of ANDERSON which generalizes the famous SERRE-Problem solved by QUILLEN and SUSLIN.

Theorem (GUBELADZE)**.** *Let Y be a normal affine toroidal variety. Then every vector bundle on Y is trivial.*

In our situation the action of the standard torus $T := (\mathbb{C}^*)^n$ on $W(= \mathbb{C}^n)$ commutes with G and defines actions of T on $W/\!\!/G$ and $W'/\!\!/G$ with dense orbits. Therefore, every vector bundle on the quotients $W/\!\!/G$ and $W'/\!\!/G$ is trivial. Since $W' \to W'/\!\!/G$ is a principal bundle it follows from [Kr89, §2, Proposition 2] that every G-vector bundle on W' is trivial, too. This proves Claim 1.

Claim 2. Let $r \leq n$ and assume that $\mathcal{V}$ restricted to $H_1, H_2, \ldots, H_r$ is trivial. Then $\mathcal{V}$ restricted to $\cup_{i=1}^r H_i$ is trivial.

Define $\Delta_i := x_1 x_2 \cdots x_i$ and $X_i := \{\Delta_i = 0\} = H_1 \cup H_2 \cup \ldots \cup H_i$. Then $X_i = X_{i-1} \cup H_i$. By induction on i, we may assume that $\mathcal{V}|_{X_i}$ is trivial for all $i < r$, and by assumption we know that $\mathcal{V}|_{H_r}$ is trivial. Choosing isomorphisms

$$\varphi\colon \mathcal{V}|_{X_{r-1}} \xrightarrow{\sim} X_{r-1} \times V \quad \text{and} \quad \psi\colon \mathcal{V}|_{H_r} \xrightarrow{\sim} H_r \times V$$

we obtain a G-automorphism $\psi \circ \varphi^{-1}$ of the trivial bundle $\Theta_V := V \times W \xrightarrow{\text{pr}} W$ defined over $X_{r-1} \cap H_r$. Consider the projection $\text{pr}_r\colon W \to H_r$ with kernel $\mathbb{C} \cdot w_r$. It induces an equivariant projection $\pi_r\colon X_{r-1} \to X_{r-1} \cap H_r$ which is the identity on $X_{r-1} \cap H_r$. Then $(\psi \circ \varphi^{-1}) \circ \pi_r$ is an automorphism of Θ_V over X_{r-1}. Changing φ by this automorphism, we can arrange that φ and ψ agree on $X_{r-1} \cap H_r$. Since X_{r-1} and H_r intersect transversally, φ and ψ glue together to give a trivialization of $\mathcal{V}$ over X_r. This proves Claim 2.

Let X denote $\bigcup_{i=1}^n H_i$. By Claim 2 and induction on $\dim W$, we have an isomorphism of $\mathcal{V}|_X$ and $\Theta_V|_X$. This implies that there is a neighborhood $U := \{h \neq 0\} \subset W$ of X, where $h \in \mathcal{O}(W)^G$, such that $\mathcal{V}$ is trivial over U (see [Kr89, §6, Proposition 6]):

$$\mathcal{V}|_U \xrightarrow{\sim} U \times V.$$

Moreover, we already know that $\mathcal{V}|_{W'}$ is trivial (Claim 1). Thus $\mathcal{V}$ is defined by a G-equivariant "glueing function" $S \in \text{Mor}(U \cap W', \text{GL}(V))^G$. Of course, G-invariance means that $S(w) = \rho(g)^{-1}(S(gw))\rho(g)$ for $w \in U \cap W'$, where $\rho\colon G \to \text{GL}(V)$ is the given G-action on V.

Claim 3. We may assume that χ_1 is the trivial character and that there are monomials $d_i \in \mathbb{C}[x_1, x_1^{-1}, \ldots, x_n^{-1}]$ such that d_i has weight λ_i, $i = 1, \ldots, m$.

Let Ω denote the lattice of weights of $\mathcal{O}(W') = \mathbb{C}[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]$. We may arrange that for some r, the first r characters $\lambda_1, \ldots, \lambda_r$ are pairwise non-equivalent modulo Ω and that each λ_j is equivalent to one of $\lambda_1, \ldots, \lambda_r$ modulo Ω. Then we may write $V = \oplus_{i=1}^r V_i$, where V_i is the sum of the weight spaces with weights equivalent to λ_i modulo Ω. By definition of the V_i, $S(w)(V_i) \subset V_i$ for all $w \in W'$. Thus $\mathcal{V}$ breaks up into a direct sum $\mathcal{V}_1, \dots, \mathcal{V}_r$, and we can reduce to the case that $\mathcal{V} = \mathcal{V}_1$, i.e., that all the λ_i are equivalent modulo Ω. Clearly, $\mathcal{V}$ is trivial if and only if it is trivial when multiplied by a character of G. Thus we may assume that λ_1 is the trivial character, and therefore that all the λ_i belong to Ω. We have Claim 3.

Let D be the diagonal matrix with diagonal entries $d_1, d_2, \dots, d_m$. Then $D \in \operatorname{Mor}(W', \operatorname{Iso}(\mathbb{C}^m, V))^G$, where $\mathbb{C}^m$ is given the trivial G-action. Thus $\tilde{S} = D^{-1}SD \in \operatorname{Mor}(U \cap W', \operatorname{GL}_m(\mathbb{C}))^G$. Now $\tilde{S}$ can be considered as the "glueing function" for a vector bundle on the quotient $W/\!\!/G$. By Gubeladze's Theorem this bundle is trivial, so $\tilde{S}$ can be written as a product $\tilde{S} = AB$ where $A \in \operatorname{Mor}(U, \operatorname{GL}_m(\mathbb{C}))^G$ and $B \in \operatorname{Mor}(W', \operatorname{GL}_m(\mathbb{C}))^G$. From this we obtain that

$$S = (DAD^{-1})\ (DBD^{-1}).$$

Clearly, $DBD^{-1} \in \operatorname{Mor}(W', \operatorname{GL}(V))^G$. Now the ij-entry of DAD^{-1} is $d_i d_j^{-1} A_{ij}$. Hence $DAD^{-1} \in \operatorname{Mor}(U, \operatorname{GL}(V))^G$ (finishing our proof) if the off diagonal entries A_{ij} are congruent to $0 \mod \Delta^N$ for large enough N, where $\Delta = x_1 \cdots x_n$. The claim below shows that there is an element $\tilde{A} \in \operatorname{Mor}(W, \operatorname{GL}_m(\mathbb{C}))^G$ such that $(A\tilde{A})_{ij} \equiv 0 \mod \Delta^N$, $i \neq j$. Thus we may write $\tilde{S} = (A\tilde{A})(\tilde{A}^{-1}B)$, where now $D(A\tilde{A})D^{-1} \in \operatorname{Mor}(U, \operatorname{GL}(V))^G$, finishing the proof of the Theorem.

Claim 4. For any $N \geq 0$ there is an $\tilde{A} \in \operatorname{Mor}(W, \operatorname{GL}_m(\mathbb{C}))^G$ such that, for all $i \neq j$, $(A\tilde{A})_{ij} \equiv 0 \mod \Delta^N$.

Since $U = \{h \neq 0\}$, if we can find an $\tilde{A}$ which "works" for $h^q A$ for some $q \in \mathbb{N}$, then it also works for A. Hence we may assume that $A \in \operatorname{Mor}(W, \operatorname{M}_m(\mathbb{C}))^G$. Now $A|_{H_1}$ is invertible, hence $A_1 := (A|_{H_1}) \circ \operatorname{pr}_1$ is invertible, where $\operatorname{pr}_1 \colon W \to H_1$ is the equivariant projection with kernel $\mathbb{C}\cdot w_1$. Thus $AA_1^{-1}|_{H_1}$ is the identity. Now consider the equivariant projection $\operatorname{pr}_2 \colon W \to H_2$ with kernel $\mathbb{C}\cdot w_2$ and the induced projection $\pi_2 \colon H_1 \to H_1 \cap H_2$. Then the matrix $A_2 := (AA_1^{-1}) \circ \operatorname{pr}_2$ satisfies the condition $A_2|_{H_1} = (A_2|_{H_1 \cap H_2}) \circ \pi_2 = I$, and so $AA_1^{-1}A_2^{-1}|_{H_1 \cup H_2} = I$, etc. Therefore we may assume that $A|_X$ is the identity, i.e., that $A \equiv I \mod \Delta$.

By induction we may suppose that $A_{ij} \equiv 0 \mod \Delta^N$, $i \neq j$. So A has the form

$$A = \begin{pmatrix} 1 + \Delta f_{11} & \Delta^N f_{12} & \cdots & \Delta^N f_{1m} \\ \Delta^N f_{21} & 1 + \Delta f_{22} & \cdots & \Delta^N f_{2m} \\ \vdots & \vdots & \ddots & \\ \Delta^N f_{m1} & \Delta^N f_{m2} & \cdots & 1 + \Delta f_{mm} \end{pmatrix}.$$

By elementary column operations, i.e., by subtracting the $\Delta^N f_{1i}$ multiple of the first column from the i-th column we get a matrix of the form

$$A' = \begin{pmatrix} 1+\Delta f_{11} & \Delta^{N+1} f'_{12} & \cdots & \Delta^{N+1} f'_{1m} \\ \Delta^N f_{21} & 1+\Delta f'_{22} & \cdots & \Delta^N f'_{2m} \\ \vdots & \vdots & \ddots & \\ \Delta^N f_{m1} & \Delta^N f'_{m2} & \cdots & 1+\Delta f'_{mm} \end{pmatrix}.$$

Continuing with the second column and so on we finally obtain a matrix whose off-diagonal elements are all $\equiv 0 \mod \Delta^{N+1}$. The elementary column operations used above correspond to right multiplication by some $\tilde{A} \in \mathrm{Mor}(W, \mathrm{SL}_m(\mathbb{C}))^G$, so Claim 4 is established. □

Corollary. *Let $G = G_1 \times G_2$ be a reductive group where G_2 is commutative. Let W be a representation space of G where G_1 acts trivially. Then every G-vector bundle over W is trivial.*

Proof. Any G-vector bundle E on W breaks up into a direct sum $\bigoplus_i E_i$ where each E_i is of the form $V_i \otimes \mathcal{V}_i$ for $\mathcal{V}_i$ a G_2-vector bundle over W and V_i a representation space of G_1. This follows immediately from [Kr89, §2] (see 2.1 Proposition 2 and its proof). Now apply Theorem 3 to the $\mathcal{V}_i$. □

Note that if G is connected reductive and (G, G) acts trivially on W, then a finite cover of G has a decomposition as in the corollary, hence all G-vector bundles on W are trivial.

Remark. The proof of Claim 4 included the following result of independent interest.

Let W be a representation space of G, and let $\ell_1, \ldots, \ell_r \in W^*$ be linearly independent semi-invariants. For any representation $\rho: G \to \mathrm{GL}_m(\mathbb{C})$, the canonical homomorphism $\mathrm{Mor}(V, GL_m(\mathbb{C}))^G \to \mathrm{Mor}(X, GL_m(\mathbb{C}))^G$ is surjective, where X is the zero set of the ℓ_i.

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