

INTRODUCTION TO LIE GROUPS

Exercice sheet 1

(due on Monday, November 14th)

(1) Let X be a locally compact space. Prove that

$$\text{Homeo}(X) := \{f : X \rightarrow X : f \text{ is a homeomorphism}\}$$

is a topological group with the compact-open topology.

(2) Prove that $\text{Homeo}(S^2)$ is not locally compact.

(3) Let $B_p : \mathbb{R}^{p+q} \times \mathbb{R}^{p+q} \rightarrow \mathbb{R}$ be the bilinear nondegenerate symmetric form

$$B_p(x, y) := - \sum_{i=1}^p x_i y_i + \sum_{j=p+1}^{p+q} x_j y_j.$$

Prove that the orthogonal group of B

$$O(p, q) := \{g \in \text{GL}(p+q, \mathbb{R}) : B_p(gx, gy) = B_p(x, y), \forall x, y \in \mathbb{R}^{p+q}\}$$

is compact if and only if $p = 0$.

(4) Let \mathcal{H} be a complex separable Hilbert space and $\mathcal{U}(\mathcal{H})$ be the group of linear unitary operators on \mathcal{H} , that is the linear operators $U : \mathcal{H} \rightarrow \mathcal{H}$ which preserve the inner product on \mathcal{H} . Show that $\mathcal{U}(\mathcal{H})$ is locally compact if and only if \mathcal{H} is finite dimensional.

(5) Use the fact that over \mathbb{C} all bilinear forms B_p are equivalent to show that there is an embedding of $O(p, q)$ into $O(p+q, \mathbb{C})$.

(6) Let $M(n, \mathbb{R})$ be the vector space of $n \times n$ matrices with real coefficients. If $\det : M(n, \mathbb{R}) \rightarrow \mathbb{R}$ is the determinant homomorphism and $\text{tr} : M(n, \mathbb{R}) \rightarrow \mathbb{R}$ is the trace, show that $d \det_I = \text{tr}$, where I is the identity matrix.

(7) Let M, M' be smooth manifolds and $\varphi : M \rightarrow M'$ a diffeomorphism. Denote by $\text{Vect}(M)$ the Lie algebra of vector fields on M with the Lie bracket and define $\varphi_* : \text{Vect}(M) \rightarrow \text{Vect}(M')$ by

$$\varphi_*(X)_{m'} := d\varphi_{\varphi^{-1}(m')} X_{\varphi^{-1}(m')},$$

for all $X \in \text{Vect}(M)$. Show that φ_* is a homomorphism of Lie algebras, that is for all $X, Y \in \text{Vect}(M)$

$$\varphi_*[X, Y] = [\varphi_*(X), \varphi_*(Y)].$$